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Gauge fixing and coBRST

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Abstract

It has previously been shown that a BRST quantization on an inner product space leads to physical states of the form

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle$$

where $|\phi\rangle$ is either a trivially BRST invariant state which only depends on the matter variables, $|\phi\rangle_1$, or a solution of a Dirac quantization, $|\phi\rangle_2$. ψ is a corresponding fermionic gauge fixing operator, ψ_1 or ψ_2. We show here for abelian and nonabelian models that one may also choose a linear combination of ψ_1 and ψ_2 for both choices of $|\phi\rangle$ except for a discrete set of relations between the coefficients. A general form of the coBRST charge operator is also determined and shown to be equal to such a ψ for an allowed linear combination of ψ_1 and ψ_2. This means that the coBRST charge is always a good gauge fixing fermion.

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1 Introduction.

In a BRST quantization one starts from a BRST invariant theory defined on a non-degenerate inner product space, V, and projects out the BRST invariant states. Of particular interest are the BRST singlets, $|s\rangle$, which represent the true physical states ($|s\rangle \in \text{Ker}Q/\text{Im}Q$). They may be chosen to be orthogonal to all unphysical states in V. In ref.[1] it was shown that within the operator formulation of general BRST invariant theories with finite number of degrees of freedom the BRST singlet states on inner product spaces may be represented in the form

$$|s\rangle = e^{[Q,\psi]}|\phi\rangle \tag{1.1}$$

where Q is the hermitian, nilpotent BRST charge, ψ a hermitian fermionic gauge fixing operator and $|\phi\rangle$ a BRST invariant state vector which does not belong to an inner product space. More precisely it was shown that there exist two sets of hermitian operators each consisting of BRST doublets in involution, *i.e.* we have

$$D_{(l)r} = \{C_{(l)i}, B_{(l)i}\}, \quad l = 1, 2 \tag{1.2}$$

where (discarding factors of i)

$$B_{(l)i} = [Q, C_{(l)i}] \tag{1.3}$$

which satisfy

$$[D_{(l)r}, D_{(l)s}] = K_{(l)rs}{}^{t} D_{(l)t} = D_{(l)t} K^{\dagger}_{(l)sr}{}^{t}, \quad l = 1, 2 \tag{1.4}$$

(We use graded commutators [1].) In addition, these sets also satisfy the condition

$$[D_{(1)r}, D_{(2)s}] \text{ is an invertible matrix operator.} \tag{1.5}$$

The doublets (1.2) determine two equivalent forms of formula (1.1), namely

$$|s\rangle_l = e^{[Q,\psi_l]}|\phi\rangle_l, \quad l = 1, 2 \tag{1.6}$$

where the states $|\phi\rangle_l$ satisfy the conditions

$$D_{(l)r}|\phi\rangle_l = 0, \quad l = 1, 2 \tag{1.7}$$

which are consistent due to (1.4), and which always imply that $|\phi\rangle_l$ are BRST invariant. For the states $|s\rangle_l$ we have then

$$D'_{(l)r}|s\rangle_l = 0, \quad l = 1, 2 \tag{1.8}$$

where

$$D'_{(l)} \equiv e^{[Q,\psi_l]} D_{(l)} e^{-[Q,\psi_l]} \tag{1.9}$$

If

$$[D'_{(l)r}, (D'_{(l)s})^{\dagger}] \text{ is an invertible matrix operator} \tag{1.10}$$

then $D'_{(l)r}$ and $(D'_{(l)s})^\dagger$ constitute BRST quartets and $|s\rangle_l$ are singlet states due to the quartet mechanism [2, 3]. Condition (1.10) determines the allowed class of hermitian gauge fixing fermions ψ_l in (1.6). In [1] it was shown that an allowed choice is

$$\psi_1 = C^{(b)}_{(2)a} C^{(f)a}_{(2)}, \quad \psi_2 = C^{(b)}_{(1)a} C^{(f)a}_{(1)} \tag{1.11}$$

where $C^{(b)a}$ and $C^{(f)a}$ are the bosons and fermions respectively of the C-operators in the BRST doublets (1.2) which are assumed to commute in (1.11). (Notice that the condition (1.5) requires the C-operators to consist of equally many bosons and fermions.)

When these results were applied to general, both irreducible and reducible gauge theories of arbitrary rank within the BFV formulation in [1] it was shown that there always exists a rather simple representation of the two sets of BRST doublets $D_{(l)}$ which makes (1.7) simple to solve. For instance, for an arbitrary irreducible gauge theory we have the BFV-BRST charge operator

$$Q = \mathcal{C}^a \theta_a + \bar{\mathcal{P}}^a \pi_a + \dots \tag{1.12}$$

where $\mathcal{C}^a$ are ghost operators, $\bar{\mathcal{P}}^a$ conjugate momenta to the antighosts $\bar{\mathcal{C}}^a$, and π_a the conjugate momenta to the Lagrange multipliers v^a. θ_a are the gauge generators which are in involution. The dots in (1.12) represent terms involving the ghosts $\mathcal{C}^a$ and their conjugate momenta $\mathcal{P}_a$. They are determined by the condition $Q^2 = 0$ and the precise form of the commutator $[\theta_a, \theta_b]$ (see *e.g.* refs [4, 5]). (For abelian gauge theories, $[\theta_a, \theta_b] = 0$, the first two terms in (1.12) are sufficient.) In this case the BRST doublets, $D_{(l)}$, are naturally given by (apart from factors of i) [1]

$$\begin{aligned} D_{(1)} &= \{\chi^a, [Q, \chi^a]; \bar{\mathcal{C}}_a, \pi_a\} \\ D_{(2)} &= \{v^a, \bar{\mathcal{P}}^a; \mathcal{P}_a, [Q, \mathcal{P}_a] = \theta_a + \dots\} \end{aligned} \tag{1.13}$$

where χ^a are gauge fixing conditions to θ_a which are required to be in involution. Conditions (1.7) imply then that $|\phi\rangle_1$ is a ghost fixed state which does not depend on the Lagrange multipliers and the gauge generators, while $|\phi\rangle_2$ satisfies a Dirac quantization apart from also being a ghost fixed state which does not depend on the Lagrange multipliers. The corresponding gauge fixing fermions may be chosen to be

$$\psi_1 = \mathcal{P}_a v^a, \quad \psi_2 = \bar{\mathcal{C}}_a \chi^a \tag{1.14}$$

One may notice that the two singlets

$$|s\rangle_l = e^{[Q, \psi_l]} |\phi\rangle_l, \quad l = 1, 2 \tag{1.15}$$

are BRST invariant states even if $|\phi\rangle_l$ does not satisfy the following gauge fixing conditions

$$\chi^a |\phi\rangle_1 = 0, \quad v^a |\phi\rangle_2 = 0 \tag{1.16}$$

In ref.[6, 7, 8] the expressions (1.15) without the conditions (1.16) were obtained by means of a bigrading in the case when the gauge group is a general Lie group.

In this paper we present two generalizations of the results of refs [1, 6, 7, 8]. First we consider the possibility to generalize the form of the gauge fixing fermions (1.14). Remember that in the conventional treatment of the BFV-theory the gauge fixing fermions,

which there enter into the Hamiltonians, are usually chosen to be a linear combination of ψ_1 and ψ_2 in (1.14) (see *e.g.* ref. [4]). Since these gauge fixing fermions are equal to ours apart from a multiplicative time parameter according to ref.[9] we should also be able to use gauge fixing fermions which are linear combinations of ψ_1 and ψ_2 in (1.15). Indeed, in sections 3 and 6 we prove for abelian respectively nonabelian models that (1.6) are still singlet states as well as inner product states when ψ_l is an arbitrary linear combination of those in (1.11) and when $|\phi\rangle_l$ satisfy (1.7) *except for a discrete set of relations between the coefficients of* ψ_1 *and* ψ_2.

The second subject of this paper concerns the role of coBRST invariance in the above construction. In ref [1] it was suggested that the coBRST charge operator, ${}^*Q$, should provide for a more invariant formulation. Indeed, the conditions

$$Q|s\rangle = {}^*Q|s\rangle = 0 \tag{1.17}$$

do project out singlet states from an original nondegenerate inner product space V [3, 10, 11, 12]. The coBRST charge is defined by

$${}^*Q \equiv \eta Q \eta \tag{1.18}$$

where η is a hermitian metric operator which maps the original state space V onto a Hilbert space. It satisfies

$$\begin{aligned} &\langle u|\eta|u\rangle \geq 0, \;\; \forall |u\rangle \in V, \quad \eta^2 = \mathbf{1} \\ &(\langle u|\eta|u\rangle = 0 \;\Leftrightarrow\; |u\rangle = 0) \end{aligned} \tag{1.19}$$

The coBRST charge ${}^*Q$ is simply the hermitian conjugate of Q in this Hilbert space. Notice that ${}^*Q$ is also nilpotent. In terms of the coBRST charge we have the Hodge decomposition, which means that any state $|u\rangle \in V$ may uniquely be written as

$$|u\rangle = |s\rangle + Q|u_1\rangle + {}^*Q|u_2\rangle, \tag{1.20}$$

where the singlet states $|s\rangle$ are determined by (1.17) or equivalently

$$\triangle|s\rangle = 0, \quad \triangle \equiv [Q, {}^*Q] \tag{1.21}$$

(see *e.g.* [12]). One may also show that every state in the non-physical space can be written as a linear combination of eigenstates of the $\triangle$ operator. The eigenvalues corresponding to these eigenstates are positive real numbers. ($\triangle$ is hermitian in the Hilbert space: ${}^*\triangle = \eta\triangle\eta = \triangle$.)

In section 2 we summarize known results for abelian models and in section 3 we derive the conditions for singlet states of the form (1.6) with generalized gauge fixing fermions. In section 4 we connect the form (1.6) with a general Fock space construction, and in section 5 we construct η and the coBRST charge ${}^*Q$ for the simple abelian model of sections 2 and 3. The BFV form of ${}^*Q$ turns out to have the form of an allowed gauge fixing fermion ψ. The relation between the gauge fixing fermion in (1.6) and the coBRST charge operator that annihilates this state is also given. In section 6 these results are generalized to a class of nonabelian models. In section 7 we then summarize our results and give some concluding remarks. In three appendices we give proofs of formulas and some unitary transformations used in the text.

2 A simple abelian model

In the following we shall make extensive use of a very simple abelian model whose hermitian BRST charge operator is given by

$$Q = \mathcal{C}^a p_a + \bar{\mathcal{P}}^a \pi_a \tag{2.1}$$

where p_a and π_a are hermitian conjugate momenta to the hermitian coordinates x^a and v^a respectively, and $\mathcal{C}^a$ and $\bar{\mathcal{P}}^a$ are hermitian fermionic operators conjugate to $\mathcal{P}_a$ and $\bar{\mathcal{C}}_a$ respectively. The index $a = 1, \ldots, n < \infty$ is assumed to be raised and lowered by a real symmetric metric g_{ab}. The fundamental nonzero commutators are

$$[x^a, p_a]_- = i\delta^a_b, \quad [v^a, \pi_b]_- = i\delta^a_b, \quad [\mathcal{C}^a, \mathcal{P}_b]_+ = \delta^a_b, \quad [\bar{\mathcal{C}}^a, \bar{\mathcal{P}}_b]_+ = \delta^a_b \tag{2.2}$$

One may think of (2.1) as the BRST charge operator of an abelian bosonic gauge theory where p_a are the gauge generators, v^a the Lagrange multipliers, and $\mathcal{C}^a$ and $\bar{\mathcal{C}}_a$ the ghosts and antighosts respectively. Alternatively one may view it as the BRST charge of a fermionic gauge theory with bosonic ghosts p_a and antighosts v^a, or a mixture of these two interpretations.

Applying the rules of ref.[1] as described in the introduction we find here the two dual sets $D_{(l)}$ in (1.2) to be given by (cf (1.13))

$$D_{(1)r} = \{x^a, \mathcal{C}^a, \bar{\mathcal{C}}_a, \pi_a\}, \quad D_{(2)r} = \{v^a, \bar{\mathcal{P}}^a, \mathcal{P}_a, p_a\} \tag{2.3}$$

which obviously satisfy (1.4) and (1.5). Thus, we obtain the singlet states

$$|s\rangle_l = e^{[Q,\psi_l]}|\phi\rangle_l, \quad l = 1, 2 \tag{2.4}$$

where we may *e.g.* choose the gauge fermions

$$\psi_1 = \alpha \mathcal{P}_a v^a, \quad \psi_2 = \beta \bar{\mathcal{C}}_a x^a \tag{2.5}$$

for arbitrary finite nonzero real constants α and β, and where $|\phi\rangle_1$ and $|\phi\rangle_2$ satisfy

$$\begin{aligned} x^a|\phi\rangle_1 = \mathcal{C}^a|\phi\rangle_1 = \bar{\mathcal{C}}_a|\phi\rangle_1 = \pi_a|\phi\rangle_1 = 0 \\ v^a|\phi\rangle_2 = \bar{\mathcal{P}}^a|\phi\rangle_2 = \mathcal{P}_a|\phi\rangle_2 = p_a|\phi\rangle_2 = 0 \end{aligned} \tag{2.6}$$

A more general allowed choice than (2.5) is

$$\psi_1 = \mathcal{P}_a T^{ab} v^b, \quad \psi_2 = \bar{\mathcal{C}}_a S^{ab} x^b \tag{2.7}$$

where T^{ab} and S^{ab} are real, invertible matrices.

For (2.5) the conditions (2.6) imply

$$D'_{(1)}|s\rangle_1 = 0, \quad D'_{(2)}|s\rangle_2 = 0 \tag{2.8}$$

where

$$\begin{aligned} D'_{(1)} &= \{x^a + i\alpha v^a, \mathcal{C}^a + i\alpha\bar{\mathcal{P}}^a, \bar{\mathcal{C}}_a - i\alpha\mathcal{P}_a, \pi_a - i\alpha p_a\} \\ D'_{(2)} &= \{v^a + i\beta x^a, \bar{\mathcal{P}}^a + i\beta\mathcal{C}^a, \mathcal{P}_a - i\beta\bar{\mathcal{C}}_a, p_a - i\beta\pi_a\} \end{aligned} \tag{2.9}$$

If α and β are different from zero and finite in (2.5), the condition (1.10) is satisfied, *i.e.* we have that $[D'_{(l)r}, (D'_{(l)s})^\dagger]$ is an invertible matrix operator for both $l=1$ and $l=2$ and $|s\rangle_l$ are singlet states. They are then also inner product states: In [7] it was explicitly shown that ${}_l\langle s|s\rangle_l = {}_l\langle\phi|e^{2[Q,\psi_l]}|\phi\rangle_l$ are finite for the above model if α and β are finite and non-zero. In fact, for the more general choice (2.7) we have

$$ {}_1\langle s|s\rangle_1 = \frac{\det T}{|\det T|}A, \quad {}_2\langle s|s\rangle_2 = \frac{\det S}{|\det S|}B \tag{2.10} $$

where A and B are finite expressions. (Thus, if α or β is zero or infinite in (2.5) we have the badly defined expressions $\langle s|s\rangle = 0/0$ or $\langle s|s\rangle = \infty/\infty$!)

3 Generalized gauge fixing for the abelian model

Here we investigate under what conditions states of the form

$$ |s\rangle_l = e^{[Q,\psi]}|\phi\rangle_l, \quad l=1,2 \tag{3.1} $$

are singlet states for the abelian model where the gauge fixing fermion ψ is a linear combination of those in (2.5), *i.e.*

$$ \psi = \alpha\mathcal{P}_a v^a + \beta\bar{\mathcal{C}}_a x^a \tag{3.2} $$

(The states $|\phi\rangle_1$ and $|\phi\rangle_2$ are still required to satisfy the conditions in (2.6).) Thus, (3.1) is a generalization of (2.4). The conditions (2.6) imply now

$$ D'_{(l)r}|s\rangle_l = 0, \quad l=1,2 \tag{3.3} $$

where $D'_{(l)}$ is given by

$$ D'_{(l)} = e^{[Q,\psi]}D_{(l)}e^{-[Q,\psi]} \tag{3.4} $$

For $\alpha\beta > 0$ we find $D'_{(1)r}$ to contain

$$ \begin{aligned} x'^a &\equiv e^{[Q,\psi]}x^a e^{-[Q,\psi]} = x^a \cos\sqrt{\alpha\beta} + i\alpha v^a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ \mathcal{C}'^a &\equiv e^{[Q,\psi]}\mathcal{C}^a e^{-[Q,\psi]} = \mathcal{C}^a \cos\sqrt{\alpha\beta} + i\alpha\bar{\mathcal{P}}^a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ \bar{\mathcal{C}}'_a &\equiv e^{[Q,\psi]}\bar{\mathcal{C}}_a e^{-[Q,\psi]} = \bar{\mathcal{C}}_a \cos\sqrt{\alpha\beta} - i\alpha\mathcal{P}_a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ {\pi'}_a &\equiv e^{[Q,\psi]}\pi_a e^{-[Q,\psi]} = \pi_a \cos\sqrt{\alpha\beta} - i\alpha p_a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \end{aligned} \tag{3.5} $$

and $D'_{(2)r}$

$$ \begin{aligned} v'^a &\equiv e^{[Q,\psi]}v^a e^{-[Q,\psi]} = v^a \cos\sqrt{\alpha\beta} + i\beta x^a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ \bar{\mathcal{P}}'^a &\equiv e^{[Q,\psi]}\bar{\mathcal{P}}^a e^{[-Q,\psi]} = \bar{\mathcal{P}}^a \cos\sqrt{\alpha\beta} + i\beta\mathcal{C}^a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ {\mathcal{P}'}_a &\equiv e^{[Q,\psi]}\mathcal{P}_a e^{-[Q,\psi]} = \mathcal{P}_a \cos\sqrt{\alpha\beta} - i\beta\bar{\mathcal{C}}_a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \\ {p'}_a &\equiv e^{[Q,\psi]}p_a e^{-[Q,\psi]} = p_a \cos\sqrt{\alpha\beta} - i\beta\pi_a \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \end{aligned} \tag{3.6} $$

For $\alpha\beta < 0$ we get the same expressions with the replacements

$$\cos\sqrt{\alpha\beta} \rightarrow \cosh\sqrt{-\alpha\beta}, \quad \frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \rightarrow \frac{\sinh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}} \tag{3.7}$$

In order to satisfy (1.10) we must have

$$[x'^a, (\pi'_b)^\dagger]_- \text{ and } [\mathcal{C}'^a, (\bar{\mathcal{C}}'_b)^\dagger]_- \text{ are invertible} \tag{3.8}$$

for $D'_{(1)r}$, and

$$[v'^a, (p'_b)^\dagger]_+ \text{ and } [\bar{\mathcal{P}}'^a, (\mathcal{P}'_b)^\dagger]_+ \text{ are invertible} \tag{3.9}$$

for $D'_{(2)r}$. For (3.5) and (3.6) we find explicitly

$$\begin{aligned}
&[x'^a, (\pi'_b)^\dagger]_- = -\alpha\frac{\sin 2\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}\delta^a_b, \quad [\mathcal{C}'^a, (\bar{\mathcal{C}}'_b)^\dagger]_- = i\alpha\frac{\sin 2\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}\delta^a_b \\
&[v'^a, (p'_b)^\dagger]_+ = -\beta\frac{\sin 2\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}\delta^a_b, \quad [\bar{\mathcal{P}}'^a, (\mathcal{P}'_b)^\dagger]_+ = i\beta\frac{\sin 2\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}\delta^a_b
\end{aligned} \tag{3.10}$$

Thus, for $\alpha\beta > 0$ eq.(3.8) is satisfied provided $\alpha \neq 0$ and $\sqrt{\alpha\beta} \neq n\frac{\pi}{2}$ where n is a positive integer, and eq.(3.9) is satisfied provided $\beta \neq 0$ and $\sqrt{\alpha\beta} \neq n\frac{\pi}{2}$. For $\alpha\beta < 0$ we have to make the replacement (3.7) on the right-hand sides of (3.10). This implies that (3.8) and (3.9) are then satisfied for $\alpha \neq 0$ and $\beta \neq 0$ respectively. This is true even in the limit $\alpha\beta \rightarrow 0$ in which case (3.5) and (3.6) reduce to (2.9).

We conclude that $|s\rangle = e^{[Q,\psi]}|\phi\rangle$ are singlet states for the gauge fixing (3.2) provided $\alpha \neq 0$ ($\beta \neq 0$) when $|\phi\rangle$ is chosen to satisfy the conditions of $|\phi\rangle_1$ ($|\phi\rangle_2$) in (2.6). In addition, we must have $\sqrt{\alpha\beta} \neq n\frac{\pi}{2}$ for any positive integer n when $\alpha\beta > 0$. In the path integral formulation the conditions on $|\phi\rangle$ correspond to a choice of boundary conditions [9]. Thus, when $\alpha \neq 0$, $\beta \neq 0$ and $\sqrt{\alpha\beta} \neq n\frac{\pi}{2}$ one may choose any of the two sets of conditions in (2.6) as boundary conditions.

It remains to investigate under which conditions the states (3.1) have finite norms. Let us write

$$|s\rangle_l = e^{\alpha K_1 + \beta K_2}|\phi\rangle_l, \quad l = 1, 2 \tag{3.11}$$

where we have introduced the hermitian operators

$$\begin{aligned}
K_1 &\equiv [Q, \mathcal{P}_a v^a] = v^a p_a + i\mathcal{P}_a\bar{\mathcal{P}}^a \\
K_2 &\equiv [Q, \bar{\mathcal{C}}_a x^a] = x^a \pi_a + i\bar{\mathcal{C}}_a\mathcal{C}^a
\end{aligned} \tag{3.12}$$

If we in addition introduce the hermitian operator K_3 defined by

$$\begin{aligned}
K_3 &\equiv i\frac{1}{2}[K_1, K_2]_- = \frac{1}{2}\left(v^a\pi_a - p_a x^a - i\mathcal{P}_a\mathcal{C}^a - i\bar{\mathcal{P}}^a\bar{\mathcal{C}}_a\right) = \\
&= \frac{1}{2}\left(\pi_a v^a - x^a p_a + i\mathcal{C}^a\mathcal{P}_a + i\bar{\mathcal{C}}_a\bar{\mathcal{P}}^a\right)
\end{aligned} \tag{3.13}$$

we find that the algebra of the K_i operators are closed and given by

$$[K_1, K_2] = -2iK_3, \quad [K_1, K_3] = iK_1, \quad [K_2, K_3] = -iK_2 \tag{3.14}$$

This is an SL(2,R) algebra. (By means of the identification $\phi_1 = 1/2(K_2 - K_1)$, $\phi_2 = 1/2(K_1 + K_2)$, $\phi_3 = K_3$ we obtain the standard SL(2,R) algebra $[\phi_i, \phi_j] = i\varepsilon_{ij}{}^k\phi_k$ with the metric $\mathrm{Diag}(\eta_{ij}) = (-1, +1, +1)$.) By means of the properties

$$K_2|\phi\rangle_1 = K_1|\phi\rangle_2 = K_3|\phi\rangle_1 = K_3|\phi\rangle_2 = 0 \tag{3.15}$$

it is then straight-forward to derive the following relations (a proof is given in appendix A)

$$|s\rangle_1 = e^{\alpha K_1 + \beta K_2}|\phi\rangle_1 = e^{\alpha' K_1}|\phi\rangle_1, \quad |s\rangle_2 = e^{\alpha K_1 + \beta K_2}|\phi\rangle_2 = e^{\beta' K_2}|\phi\rangle_2 \tag{3.16}$$

where

$$\alpha' = \alpha \frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}, \quad \beta' = \beta \frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \tag{3.17}$$

for $\alpha\beta > 0$ and

$$\alpha' = \alpha \frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}}, \quad \beta' = \beta \frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}} \tag{3.18}$$

for $\alpha\beta < 0$. Indeed (3.5) and (3.6) are equivalent to (2.9) with α and β replaced by α' and β'. From (3.16) it follows that provided α' and β' are non-zero and finite $|s\rangle_1$ and $|s\rangle_2$ are well defined inner product states. The conditions for this are identical to the conditions from (3.10) for $|s\rangle_1$ and $|s\rangle_2$ to be singlet states. Thus, as soon as $|s\rangle$ is a singlet state it is also an inner product state, and vice versa.

4 Fock space representation of the singlet states

In order to acquire a deeper understanding of the results of section 3 we construct here a general Fock like representation of the singlet states for the simple abelian model presented in section 2. For this purpose we introduce the complex covariant bosonic operators

$$\phi_a \equiv ap_a + b\pi_a, \quad \xi^a \equiv cv^a + dx^a \tag{4.1}$$

where a, b, c and d are complex constants. We require then

$$[\xi^a, \phi_b]_- = 0, \quad [\xi^a, \phi_b^\dagger]_- = \delta^a_b \tag{4.2}$$

from which we find

$$c = \frac{ia}{ab^* - a^*b}, \quad d = \frac{ib}{a^*b - ab^*} \tag{4.3}$$

Similarly we introduce the complex fermionic operators

$$\rho^a \equiv e\mathcal{C}^a + f\bar{\mathcal{P}}^a, \quad k_a \equiv g\mathcal{P}_a + h\bar{\mathcal{C}}_a \tag{4.4}$$

The conditions

$$[\rho^a, k_b]_+ = 0, \quad [\rho^a, k_b^\dagger]_+ = \delta^a_b \tag{4.5}$$

require here

$$g = \frac{f}{fe^* - f^*e}, \quad h = \frac{e}{f^*e - fe^*} \tag{4.6}$$

We demand now that the BRST charge (2.1) must be possible to write as

$$Q = \rho^{a\dagger}\phi_a + \phi_a^\dagger \rho^a \tag{4.7}$$

The reason is that a Q of this form requires the BRST invariant states, which contain the singlet states, to satisfy the simple conditions

$$\phi_a|ph\rangle = \rho^a|ph\rangle = 0 \tag{4.8}$$

or

$$\phi_a^\dagger|ph\rangle = \rho^{a\dagger}|ph\rangle = 0 \tag{4.9}$$

For (4.1) and (4.4) the form (4.7) leads to the additional condition

$$e = \frac{b}{ba^* - b^*a}, \quad f = \frac{a}{b^*a - ba^*} \tag{4.10}$$

which when inserted into (4.6) implies

$$g = a, \quad h = b \tag{4.11}$$

Our complex operators may then be expressed in terms of only two arbitrary constants a and b which are nonzero and subjected to the condition $b^*a - ba^* \neq 0$.

The corresponding singlet states to the physical states in (4.8) and (4.9) satisfy

$$\phi_a|s\rangle = \rho^a|s\rangle = k_a|s\rangle = \xi^a|s\rangle = 0 \tag{4.12}$$

or

$$\phi_a^\dagger|s\rangle = \rho^{a\dagger}|s\rangle = k_a^\dagger|s\rangle = \xi^{a\dagger}|s\rangle = 0 \tag{4.13}$$

where the operators $\phi_a,\ \rho^a,\ k_a,\ \xi^a,\ \phi_a^\dagger,\ \rho^{a\dagger},\ k_a^\dagger,\ \xi^{a\dagger}$ constitute two sets of BRST quartets: $(\phi_a,\ \xi^{a\dagger},\ \rho^{a\dagger},\ k_a)$ and $(\phi_a^\dagger,\ \xi,\ \rho^a,\ k_a^\dagger)$. If there are no other variables in the theory $|s\rangle$ is just a vacuum state and all the variables of the theory are unphysical.

The question is now whether or not the "vacuum" state $|s\rangle$ defined by (4.12) or (4.13) is normalizable. In order to investigate this we make a transition to a wave function representation $\psi_s(\mathcal{C}, \bar{\mathcal{C}}, x, v) = \langle \mathcal{C}, \bar{\mathcal{C}}, x, v|s\rangle$ where $\mathcal{C}^a,\ \bar{\mathcal{C}}_a,\ x^a$ and v^a are the eigenvalues of the corresponding operators. The conditions (4.12) imply then

$$\begin{aligned} 0 &= \langle \mathcal{C}, \bar{\mathcal{C}}, x, v|\phi_a|s\rangle = -i\left(a\frac{\partial}{\partial x^a} + b\frac{\partial}{\partial v^a}\right)\psi_s(\mathcal{C}, \bar{\mathcal{C}}, x, v) \\ 0 &= \langle \mathcal{C}, \bar{\mathcal{C}}, x, v|\xi^a|s\rangle = (cv^a + dx^a)\,\psi_s(\mathcal{C}, \bar{\mathcal{C}}, x, v) \end{aligned} \tag{4.14}$$

$$\begin{aligned} 0 &= \langle \mathcal{C}, \bar{\mathcal{C}}, x, v|\rho^a|s\rangle = \left(e\mathcal{C}^a + f\frac{\partial}{\partial \bar{\mathcal{C}}_a}\right)\psi_s(\mathcal{C}, \bar{\mathcal{C}}, x, v) \\ 0 &= \langle \mathcal{C}, \bar{\mathcal{C}}, x, v|k_a|s\rangle = \left(a\frac{\partial}{\partial \mathcal{C}^a} + b\bar{\mathcal{C}}_a\right)\psi_s(\mathcal{C}, \bar{\mathcal{C}}, x, v) \end{aligned} \tag{4.15}$$

Obviously these conditions allow for solutions of the form $\psi_s(\mathcal{C},\bar{\mathcal{C}},x,v)=\psi_s(\mathcal{C},\bar{\mathcal{C}})\psi_s(x,v)$ where (4.14) determines $\psi_s(x,v)$ and (4.15) $\psi_s(\mathcal{C},\bar{\mathcal{C}})$. The solution of (4.14) is

$$\psi_s(x,v)\propto\delta^n(v+\frac{d}{c}x) \tag{4.16}$$

Now the argument of the delta function must be real. If *e.g.* v^a and x^a have real eigenvalues then

$$\frac{e}{f}=\frac{d}{c}=-\frac{b}{a} \tag{4.17}$$

must be real. However, in this case we find

$$\langle s|s\rangle\propto\int d^nxd^nv\left(\delta^n(v+\frac{d}{c}x)\right)^2=\infty \tag{4.18}$$

On the other hand if one of the eigenvalues are imaginary we get a finite result: Let *e.g.* x^a have imaginary eigenvalues iu^a. The corresponding eigenstates to x^a satisfy then the relations [13, 14]

$$\begin{aligned}&x^a|iu\rangle=iu^a|iu\rangle,\quad\langle-iu|=(|iu\rangle)^\dagger\\&\langle iu'|iu\rangle=\delta^m(u'-u),\quad\int d^nu|-iu\rangle\langle-iu|=\int d^nu|iu\rangle\langle iu|=\mathbf{1}\end{aligned} \tag{4.19}$$

which implies (in this case the ratio (4.17) must be imaginary in order for the argument of the delta function (4.16) to be real)

$$\begin{aligned}&\langle s|s\rangle\propto\int d^nud^nv\psi_s^*(-iu,v)\psi_s(iu,v)=\\&=\int d^nud^nv\delta^n(v-i\frac{d}{c}u)\delta^n(v+i\frac{d}{c}u)=\left|\frac{c}{2d}\right|<\infty\end{aligned} \tag{4.20}$$

Similarly it follows that the bosonic part of $\langle s|s\rangle$ is infinite when both x^a and v^a have imaginary eigenvalues, and that it is finite also when x^a is real and v^a imaginary. For the fermionic part we get on the other hand zero for (4.17) real, and finite when it is imaginary. The same results are obtained if we use (4.13) in (4.14) and (4.15).

To conclude we have found that $\langle s|s\rangle$ is only well defined and finite when

$$\frac{e}{f}=\frac{d}{c}=-\frac{b}{a}=-ir \tag{4.21}$$

where r is a real constant which is finite and different from zero. In this case our complex operators have the form

$$\begin{aligned}&\phi_a\equiv a(p_a+ir\pi_a),\quad\xi^a\equiv\frac{1}{2a^*}(ix^a-\frac{1}{r}v^a)\\&\rho^a\equiv\frac{1}{2a^*}(\mathcal{C}^a+i\frac{1}{r}\bar{\mathcal{P}}^a),\quad k_a\equiv a(\mathcal{P}_a+ir\bar{\mathcal{C}}_a)\end{aligned} \tag{4.22}$$

Notice that if r is imaginary the complex operators (4.22) are essentially hermitian which is the reason why we found an undefined expression for $\langle s|s\rangle$ in this case. ($|s\rangle$ is then not a well defined inner product state but rather a state like $|\phi\rangle$ in sections 2 and 3.)

That either x^a or v^a should be chosen to have imaginary eigenvalues was one of the basic quantization rules found in [7]. The basic reason for this is that the complex bosonic operators ϕ_a and ξ^a span a Fock space with half positive and half indefinite metric states (see section 5).

We end this section by constructing a representation of $|s\rangle$ in the form (3.1), *i.e.*

$$|s\rangle = e^{[Q,\psi]}|\phi\rangle \tag{4.23}$$

where ψ is a gauge fixing fermion of the form (3.2), and where $|\phi\rangle$ satisfies one of the conditions in (2.6) for $|\phi\rangle_1$ or $|\phi\rangle_2$. We notice then that for $\alpha\beta > 0$ (4.12) implies

$$\phi'_a|\phi\rangle = \rho'^a|\phi\rangle = k'_a|\phi\rangle = \xi'^a|\phi\rangle = 0 \tag{4.24}$$

where

$$\begin{aligned}
\phi'_a \equiv e^{-[Q,\psi]}\phi_a e^{[Q,\psi]} = \quad & ap_a(\cos\sqrt{\alpha\beta} - \alpha r\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) + \\
& +ira\pi_a(\cos\sqrt{\alpha\beta} + \frac{\beta}{r}\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) \\
\xi'^a \equiv e^{-[Q,\psi]}\xi^a e^{[Q,\psi]} = \quad & \frac{1}{2a^*}ix^a(\cos\sqrt{\alpha\beta} + \frac{\beta}{r}\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) - \\
& -\frac{1}{2ra^*}v^a(\cos\sqrt{\alpha\beta} - \alpha r\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) \\
\rho'^a \equiv e^{-[Q,\psi]}\rho^a e^{[Q,\psi]} = \quad & \frac{\mathcal{C}^a}{2a^*}(\cos\sqrt{\alpha\beta} + \frac{\beta}{r}\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) + \\
& +i\frac{1}{2ra^*}\bar{\mathcal{P}}^a(\cos\sqrt{\alpha\beta} - \alpha r\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) \\
k'_a \equiv e^{-[Q,\psi]}k_a e^{[Q,\psi]} = \quad & a\mathcal{P}_a(\cos\sqrt{\alpha\beta} - \alpha r\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}) + \\
& +ira\bar{\mathcal{C}}_a(\cos\sqrt{\alpha\beta} + \frac{\beta}{r}\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}})
\end{aligned} \tag{4.25}$$

Hence, if $|\phi\rangle$ satisfies the condition for $|\phi\rangle_1$ in (2.6) then (4.25) requires

$$r = \frac{\sqrt{\alpha\beta}}{\alpha\tan\sqrt{\alpha\beta}} \tag{4.26}$$

If on the other hand $|\phi\rangle$ satisfies the condition for $|\phi\rangle_2$ in (2.6) then (4.25) requires

$$r = -\frac{\beta\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \tag{4.27}$$

For $\alpha\beta < 0$ we have to make use of the replacement (3.7) in (4.25). We find then correspondingly

$$r = \frac{\sqrt{-\alpha\beta}}{\alpha\tanh\sqrt{-\alpha\beta}} \tag{4.28}$$

and

$$r = -\frac{\beta\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}} \tag{4.29}$$

Since $r = 1/\alpha'$ for $|\phi\rangle_1$ and $r = -\beta'$ for $|\phi\rangle_2$ where α' and β' are given by (3.17) or (3.18) we notice that a finite nonzero r in (4.22) exactly excludes those values of α and β for which the representation (4.23) is not a singlet state and not an inner product state, *i.e.* the conditions found in section 3. Notice also that to every consistent choice of α and β there is a corresponding r and vice versa. However, there does not exist any choice of α and β for a given r for which (4.12) or (4.13) in the representation (4.22) allows for *both* choices of "boundary" conditions of $|\phi\rangle$ in (2.6).

Finally, one may notice that the expressions (3.5) or (3.6) are essentially obtained when (4.26) or (4.27) is inserted into (4.22). In particular yields $r = 1/\alpha$ and $r = -\beta$ in (4.22) essentially the two sets in (2.9).

5 The coBRST charge for the simple abelian model

In order to construct a general coBRST charge operator for the simple abelian model of section 2 using the definition (1.18) we need to construct the metric operator η in (1.19). This in turn requires us to diagonalize the oscillators ϕ, ξ, ρ and k in (4.22). Starting from a general linear ansatz we find the following expressions for diagonalized oscillators (suppressing indices)

$$\begin{aligned} a &= R^{-1}(\xi + M\phi), \quad b = UR^{-1}(\xi - M^{\dagger}\phi) \\ A &= S^{-1}(\rho + Nk), \quad B = VS^{-1}(\rho - N^{\dagger}k) \end{aligned} \tag{5.1}$$

They satisfy the commutator algebra (the non-zero part)

$$\begin{aligned} &[a_a, a_b^{\dagger}]_- = \delta_{ab}, \quad [b_a, b_b^{\dagger}]_- = -\delta_{ab}, \quad [a_a, b_b^{\dagger}]_- = 0 \\ &[A_a, A_b^{\dagger}]_+ = \delta_{ab}, \quad [B_a, B_b^{\dagger}]_+ = -\delta_{ab}, \quad [A_a, B_b^{\dagger}]_+ = 0 \end{aligned} \tag{5.2}$$

In (5.1) the vector operators a, b, A, B, ϕ and k have lower indices while ξ and ρ have upper ones. U and V are arbitrary unitary matrices and R, S, M and N are arbitrary complex invertible matrices. However, the hermitian parts of M and N are determined by R and S through the relations

$$M + M^{\dagger} = RR^{\dagger}, \quad N + N^{\dagger} = SS^{\dagger} \tag{5.3}$$

Hence, the hermitian parts of M and N are strictly positive. The oscillators in (5.1) are obviously noncovariant in general (except when the metric that raises and lowers indices is euclidean *i.e.* $g_{ab} = \pm\delta_{ab}$).

The metric operator η has now the form

$$\eta = \eta_B \eta_F \tag{5.4}$$

where [14]

$$\eta_B = \exp\,(i\pi \sum_{a=1}^{n} b_a^{\dagger} b_a), \quad \eta_F = \prod_{a=1}^{n} (1 + 2B_a^{\dagger} B_a) \tag{5.5}$$

which imply

$$\eta b_a \eta = -b_a, \quad \eta B_a \eta = -B_a \tag{5.6}$$

For the original oscillators ξ, ϕ, ρ and k this implies using (5.1) (notice that $M^{\dagger}(M+M^{\dagger})^{-1}M = M(M+M^{\dagger})^{-1}M^{\dagger}$ and $N^{\dagger}(N+N^{\dagger})^{-1}N = N(N+N^{\dagger})^{-1}N^{\dagger}$)

$$\begin{aligned}
\eta\xi\eta &= (M^{\dagger}-M)(M+M^{\dagger})^{-1}\xi + 2M(M+M^{\dagger})^{-1}M^{\dagger}\phi \\
\eta\phi\eta &= 2(M+M^{\dagger})^{-1}\xi + (M+M^{\dagger})^{-1}(M-M^{\dagger})\phi \\
\eta\rho\eta &= (N^{\dagger}-N)(N+N^{\dagger})^{-1}\rho + 2N(N+N^{\dagger})^{-1}N^{\dagger}k \\
\eta k\eta &= 2(N+N^{\dagger})^{-1}\rho + (N+N^{\dagger})^{-1}(N-N^{\dagger})k
\end{aligned} \tag{5.7}$$

Remarkably enough these expressions do not involve the matrices R, S, U and V in (5.1). All arbitrariness lies in the matrices M and N which partly are determined by R and S through the relations (5.3).

Formula (1.18) yields now the general coBRST charge operator

$$\begin{aligned}
{}^*Q \;&\equiv\; \eta Q\eta = \eta(\rho^{a\dagger}\phi_a + \phi_a^{\dagger}\rho^a)\eta = \\
&=\; 4k^{\dagger}N(N+N^{\dagger})^{-1}N^{\dagger}(M+M^{\dagger})^{-1}\xi + \\
&\quad +4\xi^{\dagger}(M+M^{\dagger})^{-1}N(N+N^{\dagger})^{-1}N^{\dagger}k + \\
&\quad +\rho^{\dagger}(N+N^{\dagger})^{-1}(N-N^{\dagger})(M+M^{\dagger})^{-1}(M-M^{\dagger})\phi + \\
&\quad +\phi^{\dagger}(M^{\dagger}-M)(M+M^{\dagger})^{-1}(N^{\dagger}-N)(N+N^{\dagger})^{-1}\rho + \\
&\quad +2\rho^{\dagger}(N+N^{\dagger})^{-1}(N-N^{\dagger})(M+M^{\dagger})^{-1}\xi + \\
&\quad +2\xi^{\dagger}(M+M^{\dagger})^{-1}(N^{\dagger}-N)(N+N^{\dagger})^{-1}\rho + \\
&\quad +2k^{\dagger}N(N+N^{\dagger})^{-1}N^{\dagger}(M+M^{\dagger})^{-1}(M-M^{\dagger})\phi + \\
&\quad +2\phi^{\dagger}(M^{\dagger}-M)(M+M^{\dagger})^{-1}N(N+N^{\dagger})^{-1}N^{\dagger}k
\end{aligned} \tag{5.8}$$

This expression satisfies

$$\begin{aligned}
\triangle \;&\equiv\; [Q, {}^*Q]_+ = [Q, k^{\dagger}N'(M')^{-1}\xi + \xi^{\dagger}(M')^{-1}N'k] = \\
&=\; \phi^{\dagger}N'(M')^{-1}\xi + k^{\dagger}N'(M')^{-1}\rho + \rho^{\dagger}(M')^{-1}N'k + \xi^{\dagger}(M')^{-1}N'\phi = \\
&=\; \xi N'(M')^{-1}\phi^{\dagger} - \rho N'(M')^{-1}k^{\dagger} - k(M')^{-1}N'\rho^{\dagger} + \phi(M')^{-1}N'\xi^{\dagger}
\end{aligned} \tag{5.9}$$

where

$$M' \equiv \frac{1}{2}\left(M+M^{\dagger}\right), \quad N' \equiv 2N(N+N^{\dagger})^{-1}N^{\dagger} \tag{5.10}$$

The properties of the matrices M and N require M' and N' to be invertible. As a consequence $\triangle|s\rangle = 0$ imply (4.12) or (4.13), which is also a consequence of $Q|s\rangle = {}^*Q|s\rangle = 0$ (see *e.g.* [12]). The original state space, V, is spanned by eigenstates to $\triangle$ with positive integers as eigenvalues. (This is at least true when M' and N' commute.) This leads to the Hodge decomposition (1.20) (see *e.g.* [12]).

Since only the first two terms in (5.8) contribute to the commutator (5.9), it is natural, and always allowed, to choose the matrices M and N to be hermitian, in which case (5.8) reduces to

$${}^*Q = k^{\dagger}NM^{-1}\xi + \xi^{\dagger}M^{-1}Nk \tag{5.11}$$

and (5.7) becomes

$$\eta\xi\eta = M\phi, \quad \eta\phi\eta = M^{-1}\xi, \quad \eta\rho\eta = Nk, \quad \eta k\eta = N^{-1}\rho \tag{5.12}$$

If we furthermore choose $N = \lambda M$ where λ is a real positive constant then ${}^*Q$ acquires the covariant form

$$
{}^*Q = \lambda \left(k_a^\dagger \xi^a + \xi^{\dagger a} k_a \right) \tag{5.13}
$$

Such expressions for coBRST are given in the literature (see *e.g.* ref.[11]).

By means of (4.22) the expression (5.13) may be rewritten in terms of the original variables. We find then

$$
{}^*Q = \lambda \left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r}\mathcal{P}_a v^a \right) \tag{5.14}
$$

where r is the real constant in (4.22). Such an expression for the coBRST charge seems not to have been given before. It differs *e.g.* from the suggestion given in [1].

The expression (5.14) when compared with (3.2) shows that the coBRST charge in this case may be viewed as a fermionic gauge fixing variable. This is very natural since both ψ and ${}^*Q$ have to do with gauge fixing. In fact, if ψ in the representation (3.1) is chosen to be the coBRST charge (5.14) we find the expressions (3.16) with

$$
\alpha' = -\frac{1}{r}\tanh\lambda, \quad \beta' = r\tanh\lambda \tag{5.15}
$$

Thus, α' and β' are then finite and non-zero for any finite and non-zero r and λ in (5.14), which means that (5.14) is always a good gauge fixing fermion.

It is natural to expect that also the more general expressions (5.8) or (5.11) should be possible to use as a gauge fixing fermion. In terms of the original variables (5.11) becomes

$$
{}^*Q = r\bar{\mathcal{C}}_a T^{ab} x^b - \frac{1}{r}\mathcal{P}_a T^{ab} v^b + \bar{\mathcal{C}}_a L^{ab} v^b + \mathcal{P}_a L^{ab} x^b \tag{5.16}
$$

where T^{ab}, L^{ab} are real matrices and T^{ab} is invertible. (If NM^{-1} in (5.11) is real then $L = 0$ and $T = NM^{-1}$.) Notice that the parameter r is related to the choice of oscillator basis and the matrices T^{ab}, L^{ab} to the choice of diagonal representation of this oscillator basis. r is more important since it is related to the choice of vacuum. Anyway, if (5.16) is chosen to be a gauge fixing fermion, then we find

$$
[Q, {}^*Q] \equiv A(T) + B(L) \tag{5.17}
$$

where

$$
\begin{aligned}
A(T) &\equiv r\pi_a T^{ab} x^b - \frac{1}{r} p_a T^{ab} v^b + ir\bar{\mathcal{C}}_a T^{ab}\mathcal{C}^b - i\frac{1}{r}\mathcal{P}_a T^{ab}\bar{\mathcal{P}}^b, \\
B(L) &\equiv \pi_a L^{ab} v^b + p_a L^{ab} x^b + i\bar{\mathcal{C}}_a L^{ab}\bar{\mathcal{P}}^b + i\mathcal{P}_a L^{ab}\mathcal{C}^b
\end{aligned} \tag{5.18}
$$

We notice now that $B(L)$ satisfies

$$
B(L)|\phi\rangle_1 = B(L)|\phi\rangle_2 = 0 \tag{5.19}
$$

and that

$$
[A(T), B(L)]_+ = 0 \tag{5.20}
$$

if the matrices T^{ab} and L^{ab} commute. In this latter case we find therefore

$$e^{[Q, {}^*Q]}|\phi\rangle_l = e^{A(T)}|\phi\rangle_l \equiv e^{[Q,\psi]}|\phi\rangle_l \tag{5.21}$$

where

$$\psi \equiv r\bar{\mathcal{C}}_a T^{ab} x^b - \frac{1}{r}\mathcal{P}_a T^{ab} v^b \tag{5.22}$$

which should be an allowed form in general since it is a linear combination of (2.6). (The condition (5.20) on the matrices T and L can probably be weakened.)

6 Generalizations to nonabelian theories

So far we have only performed a detailed analysis of simple abelian models. In order to demonstrate that our results are not special properties of such models in this section we shall consider the class of nonabelian models in which the gauge group is a general Lie group. Within the corresponding BRST invariant models the standard BFV-BRST charge is given by $(a, b, c = 1, \ldots, n < \infty)$[15]

$$Q = \theta_a \mathcal{C}^a - \frac{1}{2} i U_{bc}{}^a \mathcal{P}_a \mathcal{C}^b \mathcal{C}^c - \frac{1}{2} i U_{ab}{}^b \mathcal{C}^a + \bar{\mathcal{P}}_a \pi^a \tag{6.1}$$

where θ_a are the hermitian bosonic gauge generators (constraints) satisfying

$$[\theta_a, \theta_b]_- = iU_{ab}{}^c \theta_c \tag{6.2}$$

where $U_{ab}{}^c$ are the real structure constants. In direct analogy with what we had for the abelian models we propose now the following singlet representation for models invariant under (6.1)

$$|s\rangle_l = e^{[Q,\psi]}|\phi\rangle_l, \quad l = 1, 2 \tag{6.3}$$

where the gauge fixing fermion ψ is given by

$$\psi = \alpha \mathcal{P}_a v^a + \beta \bar{\mathcal{C}}_a x^a \tag{6.4}$$

where in turn the gauge fixing variables x^a are chosen such that they commute with all variables except the constraints θ_a and satisfy the conditions

$$[x^a, x^b] = 0, \quad [x^a, \theta_b] = iM^a{}_b \tag{6.5}$$

where $M^a{}_b$ is an invertible matrix operator. $|\phi\rangle_1$ and $|\phi\rangle_2$ satisfy here

$$\begin{aligned} &x^a|\phi\rangle_1 = \mathcal{C}^a|\phi\rangle_1 = \bar{\mathcal{C}}_a|\phi\rangle_1 = \pi_a|\phi\rangle_1 = 0 \\ &v^a|\phi\rangle_2 = \bar{\mathcal{P}}^a|\phi\rangle_2 = \mathcal{P}_a|\phi\rangle_2 = \left(\theta_a + i\frac{1}{2}U_{ab}{}^b\right)|\phi\rangle_2 = 0 \end{aligned} \tag{6.6}$$

Notice that only the last conditions on $|\phi\rangle_2$ differ from (2.6). ($U_{ab}{}^b = 0$ for unimodular gauge groups as in *e.g.* Yang-Mills.) The new conditions follow from the fact that

$$[Q, \mathcal{P}_a] = \theta_a + \theta_a^{gh}, \quad \theta_a^{gh} \equiv i\frac{1}{2}U_{ab}{}^c\left(\mathcal{P}_c\mathcal{C}^b - \mathcal{C}^b\mathcal{P}_c\right) \tag{6.7}$$

and

$$[Q,\mathcal{P}_a]|\phi\rangle_2 = \left(\theta_a + i\frac{1}{2}U_{ab}{}^b\right)|\phi\rangle_2 \tag{6.8}$$

As in section 3 we write now (6.3) in the following form

$$|s\rangle_l = e^{\alpha K_1+\beta K_2}|\phi\rangle_l, \quad l=1,2 \tag{6.9}$$

where now the hermitian operators K_1 and K_2 are given by

$$\begin{aligned} K_1 &\equiv [Q,\mathcal{P}_a v^a] = (\theta_a+\theta_a^{gh})v^a + i\mathcal{P}_a\bar{\mathcal{P}}^a \\ K_2 &\equiv [Q,\bar{\mathcal{C}}_a x^a] = x^a\pi_a + i\bar{\mathcal{C}}_a\mathcal{C}^b M^a{}_b \end{aligned} \tag{6.10}$$

(cf (3.12)). In addition, we introduce the hermitian operator K_3 defined by

$$\begin{aligned} K_3 \equiv i\frac{1}{2}[K_1,K_2]_- = \frac{1}{2}\left(\pi_a M^a{}_b v^b - (\theta_a+\theta_a^{gh})x^a\right) + \\ +\frac{1}{2}\left(i\bar{\mathcal{C}}_a M^a{}_b\bar{\mathcal{P}}_b - i\mathcal{P}_a M^a{}_b\mathcal{C}^b - \bar{\mathcal{C}}_a\mathcal{C}^b v^c[\theta_b, M^a{}_c]\right) \end{aligned} \tag{6.11}$$

In obtaining the last equality we have made use of the Jacoby identities

$$U_{cb}{}^d M^a{}_d = i[\theta_b, M^a{}_c] - i[\theta_c, M^a{}_b] \tag{6.12}$$

We notice the properties

$$K_2|\phi\rangle_1 = K_1|\phi\rangle_2 = K_3|\phi\rangle_1 = K_3|\phi\rangle_2 = 0 \tag{6.13}$$

which are identical to (3.15) which we had in the abelian case. However, in distinction to what we had there the K_i operators (6.10)-(6.11) do not satisfy a closed commutator algebra. On the other hand, in appendix C it is shown that

$$[K_2,K_3] = -iK_2, \quad [K_1,K_3]|\phi\rangle_1 = iK_1|\phi\rangle_1, \tag{6.14}$$

provided x^a are chosen to be canonical coordinates on the group manifold. ($M^a{}_b$ depends then only on x^a.) In fact, in this case K_i satisfy effectively an $SL(2,R)$ algebra on $|\phi\rangle_l$ and it is straight-forward to derive the relations (see appendix C)

$$|s\rangle_1 = e^{\alpha K_1+\beta K_2}|\phi\rangle_1 = e^{\alpha' K_1}|\phi\rangle_1, \quad |s\rangle_2 = e^{\alpha K_1+\beta K_2}|\phi\rangle_2 = e^{\beta' K_2}|\phi\rangle_2 \tag{6.15}$$

where

$$\alpha' = \alpha\,\frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}, \quad \beta' = \beta\,\frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \tag{6.16}$$

for $\alpha\beta > 0$ and

$$\alpha' = \alpha\,\frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}}, \quad \beta' = \beta\,\frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}} \tag{6.17}$$

for $\alpha\beta < 0$. Eqs.(6.15)-(6.17) are identical to (3.16)-(3.18) in the abelian case! The conditions for $|s\rangle_1$ and $|s\rangle_2$ to be inner product states should, therefore, be identical to the ones obtained in section 3 *i.e.* $\alpha' \neq 0$ and $\beta' \neq 0$ respectively. Notice, however, that

there exists no *general* proof that $|s\rangle_1 = e^{\alpha' K_1}|\phi\rangle_1$ and $|s\rangle_2 = e^{\beta' K_2}|\phi\rangle_2$ are inner product states in the nonabelian case although this is expected to be the case (see refs.[7, 8, 16] and below).

From the conditions (6.6) on $|\phi\rangle_l$ we may derive the conditions satisfied by $|s\rangle_1$ and $|s\rangle_2$ corresponding to (4.12) in the abelian case. If we restrict ourselves to the case when x^a are canonical coordinates on the group manifold then eq.(6.15) is valid and we have

$$\begin{aligned} &x'^a|s\rangle_1 = \mathcal{C}'^a|s\rangle_1 = \bar{\mathcal{C}}'_a|s\rangle_1 = \pi'_a|s\rangle_1 = 0 \\ &v''^a|s\rangle_2 = \bar{\mathcal{P}}''^a|s\rangle_2 = \mathcal{P}''_a|s\rangle_2 = \left(\theta''_a + i\frac{1}{2}U_{ab}{}^b\right)|s\rangle_2 = 0 \end{aligned} \tag{6.18}$$

where

$$\begin{aligned} x'^a \equiv e^{\alpha' K_1} x^a e^{-\alpha' K_1} = &\ e^{\alpha'\theta_a v^a} x^a e^{-\alpha'\theta_a v^a} \equiv f^a(x, -i\alpha' v) \\ \mathcal{C}'^a \equiv e^{\alpha' K_1}\mathcal{C}^a e^{-\alpha' K_1} = &\ A^a{}_b(-i\alpha' v)\mathcal{C}^b - i\alpha'(M^{-1})^a{}_b(-i\alpha' v)\bar{\mathcal{P}}^b \\ \bar{\mathcal{C}}'_a \equiv e^{\alpha' K_1}\bar{\mathcal{C}}_a e^{-\alpha' K_1} = &\ \bar{\mathcal{C}}_a + i\alpha'(M^{-1})^a{}_b(i\alpha' v)\mathcal{P}_b \\ \pi'_a \equiv e^{\alpha' K_1}\pi_a e^{-\alpha' K_1} = &\ \pi_a + i\alpha'(M^{-1})^b{}_a(i\alpha' v)\left(\theta_a + \theta_a^{gh}\right) - \\ &-\alpha'\frac{\partial}{\partial v^d}(M^{-1})^c{}_a(i\alpha' v)\mathcal{P}_c\bar{\mathcal{P}}^d \end{aligned} \tag{6.19}$$

$$\begin{aligned} &v''^a \equiv e^{\beta' K_2} v^a e^{-\beta' K_2} = v^a - i\beta' x^a \\ &\bar{\mathcal{P}}''^a \equiv e^{\beta' K_2}\bar{\mathcal{P}}^a e^{-\beta' K_2} = \bar{\mathcal{P}}^a - i\beta'\mathcal{C}^b M^a{}_b(x) \\ &\mathcal{P}''_a \equiv e^{\beta' K_2}\mathcal{P}_a e^{-\beta' K_2} = \mathcal{P}_a + i\beta'\bar{\mathcal{C}}_b M^b{}_a(x) \\ &\theta''_a \equiv e^{\beta' K_2}\theta_a e^{-\beta' K_2} = \theta_a + i\beta'\bar{\mathcal{C}}_b\mathcal{C}^c[M^b{}_c(x), \theta_a] \end{aligned} \tag{6.20}$$

Notice that $f^a(x, -i\alpha' v)$ are also canonical coordinates on the group manifold obtained by two successive transformations, one with coordinates x^a and one with $-i\alpha' v^a$. $A^a{}_b(-i\alpha' v) = (M^{-1})^a{}_c(i\alpha' v)M^c{}_b(-i\alpha' v)$ is the adjoint matrix representation of the group. Some properties of the matrices $M^a{}_b$ are given in appendix C. In deriving (6.19) we have made use of the relation

$$e^{\alpha' K_1} = e^{i\alpha'(M^{-1})^a{}_b(i\alpha' v)\mathcal{P}_a\bar{\mathcal{P}}^b} e^{\alpha'\left(\theta_a + \theta_a^{gh}\right)v^a} \tag{6.21}$$

which may be obtained from ref.[16]. The expressions for $\mathcal{C}'^a$ and π'_a were also obtained in [6] (formulas (4.5) and (4.8)). One may notice that (6.19) and (6.20) are nonlinear in distinction to the linear expressions (2.9) in the abelian case. From (6.19) and (6.20) it follows now that

$$\begin{aligned} &[x'^a, (\pi'_b)^\dagger] = \alpha'\left\{\frac{\partial f^a(x, -i\alpha' v)}{\partial(-i\alpha' v)^b} + M^d{}_c(x)(M^{-1})^c{}_b(-i\alpha' v)\frac{\partial f^a(x, -i\alpha' v)}{\partial x^d}\right\} \\ &[\mathcal{C}'^a, (\bar{\mathcal{C}}'_b)^\dagger] = -i\alpha'(M^{-1})^c{}_b(-i\alpha' v)\left(\delta^a_b + A^a{}_c(-i\alpha' v)\right) \end{aligned} \tag{6.22}$$

and

$$[v''^a, (\theta''_b)^\dagger] = \beta' M^a{}_b(x), \quad [\bar{\mathcal{P}}''^a, (\mathcal{P}''_b)^\dagger] = -2i\beta' M^a{}_b(x) \tag{6.23}$$

Since one may easily convince oneself that the matrices on the right-hand sides of (6.22) and (6.23) are invertible the conditions for $|s\rangle_1$ and $|s\rangle_2$ to be singlet states are $\alpha' \neq 0$

and $\beta' \neq 0$ respectively. These conditions are exactly the same as the ones we had in the abelian case, as well as those which were required for $|s\rangle_1$ and $|s\rangle_2$ to be inner product states.

We are now in principle able to calculate the coBRST charge in the same way as we did for the abelian models. However, since this requires us to diagonalize the "oscillators" (6.19) and (6.20) which is quite cumbersome, we shall not do that here (see below, however). Instead we shall just demonstrate that a coBRST charge of the same form as we had in the abelian case *i.e.*

$$ {}^*Q = \lambda \left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r}\mathcal{P}_a v^a \right) \tag{6.24} $$

will leave the singlet states (6.3) invariant under the expected conditions. (Below it will be proved that (6.24) actually is an appropriate coBRST charge.) To this end let us define ${}^*Q'$ and ${}^*Q''$ by

$$ \begin{aligned} {}^*Q' &\equiv e^{-\alpha' K_1}\, {}^*Q e^{\alpha' K_1} = \\ &= \lambda \left(r \left(\bar{\mathcal{C}}_a - i\alpha' (M^{-1})^b{}_a(-i\alpha' v)\mathcal{P}_b \right) f^a(x, i\alpha' v) - \frac{1}{r}\mathcal{P}_a v^a \right) \\ {}^*Q'' &\equiv e^{-\beta' K_2}\, {}^*Q e^{\beta' K_2} = \\ &= \lambda \left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r} \left(\mathcal{P}_a v^a + i\beta' \mathcal{P}_a x^a - i\beta' \bar{\mathcal{C}}_b M^b{}_a(x) v^a + \beta'^2 \bar{\mathcal{C}}_b M^b{}_a(x) x^a \right) \right) \end{aligned} \tag{6.25} $$

We have then by means of the properties $M^a{}_b(0) = \delta^a_b$, $f^a(0, y) = y^a$, and (6.6)

$$ \begin{aligned} {}^*Q|s\rangle_1 &= e^{\alpha' K_1}\, {}^*Q'|\phi\rangle_1 = e^{\alpha' K_1} \lambda \left(r\alpha'^2 \mathcal{P}_a v^a - \frac{1}{r}\mathcal{P}_a v^a \right) |\phi\rangle_1 \\ {}^*Q|s\rangle_2 &= e^{\beta' K_2}\, {}^*Q''|\phi\rangle_2 = e^{\beta' K_2} \lambda \left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r}\beta'^2 \bar{\mathcal{C}}_a x^a \right) |\phi\rangle_2 \end{aligned} \tag{6.26} $$

Hence, we have

$$ {}^*Q|s\rangle_1 = {}^*Q|s\rangle_2 = 0 \tag{6.27} $$

provided $r = \pm 1/\alpha'$ and $r = \pm\beta'$ respectively.

Remarkably enough there exists a simple abelianization of the BRST charge (6.1) which allows us to make use of all results of our analysis of abelian models also for the nonabelian models considered here. This abelianization is performed by means of x^a as canonical coordinates on the group manifold and the matrix $M^a{}_b(x)$ as follows: According to (C.6) in appendix C we may define hermitian conjugate momenta to x^a by

$$ p_a = (M^{-1})^b{}_a(x)\theta_b + i\frac{1}{2}(M^{-1})^b{}_a(x)\partial_c M^c{}_b(x) \tag{6.28} $$

We have then

$$ \theta_a = \frac{1}{2}\left(p_b M^b{}_a(x) + M^b{}_a(x) p_b \right) \tag{6.29} $$

Consider furthermore a unitary transformation which only affects η^a, $\mathcal{P}_a$, and p_a, and which is of the following form

$$ \begin{aligned} &\tilde{\mathcal{C}}^a = M^a{}_b(x)\mathcal{C}^b, \quad \tilde{\mathcal{P}}_a = (M^{-1})^b{}_a(x)\mathcal{P}_b \\ &\tilde{p}_a = p_a + i\frac{1}{2}\partial_a M^b{}_c(x)(M^{-1})^d{}_b(\mathcal{C}^c\mathcal{P}_d - \mathcal{P}_d\mathcal{C}^c) \end{aligned} \tag{6.30} $$

If one inserts (6.29) into (6.1) and replaces $\mathcal{C}^a$, $\mathcal{P}_a$, and p_a by $\tilde{\mathcal{C}}^a$, $\tilde{\mathcal{P}}_a$, and $\tilde{p}_a$ using (6.30) then one finds

$$Q = \tilde{\mathcal{C}}^a \tilde{p}_a + \pi_a \bar{\mathcal{P}}^a \tag{6.31}$$

which is the BRST charge (2.1) for an abelian model. (A similar abelianization of classical Yang-Mills was considered in [17].) In this way we may now apply all our results obtained for abelian models to the general nonabelian model (6.1). We have, thus, the representation (3.1) for the singlet states, *i.e.*

$$|s\rangle_l = e^{[Q,\psi]}|\phi\rangle_l, \quad l = 1, 2 \tag{6.32}$$

where the gauge fixing fermion ψ is given by

$$\psi = \alpha \tilde{\mathcal{P}}_a v^a + \beta \bar{\mathcal{C}}_a x^a = \alpha (M^{-1})^b{}_a(x) \mathcal{P}_b v^a + \beta \bar{\mathcal{C}}_a x^a \tag{6.33}$$

and where $|\phi\rangle_l$ satisfies (2.6) *i.e.*

$$\begin{aligned} x^a|\phi\rangle_1 = \tilde{\mathcal{C}}^a|\phi\rangle_1 = \bar{\mathcal{C}}_a|\phi\rangle_1 = \pi_a|\phi\rangle_1 = 0 \\ v^a|\phi\rangle_2 = \bar{\mathcal{P}}^a|\phi\rangle_2 = \tilde{\mathcal{P}}_a|\phi\rangle_2 = \tilde{p}_a|\phi\rangle_2 = 0 \end{aligned} \tag{6.34}$$

Since $M^a{}_b$ is an invertible matrix operator one may easily show that the conditions (6.34) are equivalent to (6.6). From our analysis of abelian models we have now that if K_2 is defined by (6.10) and

$$K_1 \equiv [Q, \mathcal{P}_a M^a{}_b(x) v^b], \quad K_3 \equiv i\frac{1}{2}[K_1, K_2] \tag{6.35}$$

then K_i will satisfy the $SL(2,R)$ algebra (3.14) exactly which was not the case above. The properties (6.13), (6.15)-(6.17) are then easily verified. This means that (6.32) are singlet states under exactly the same conditions on α and β in (6.33) as (6.3) are singlets for α and β in (6.4). From section 5 we obtain the coBRST charge of the general form (5.16), *i.e.*

$${}^*Q = \lambda \left(r\bar{\mathcal{C}}_a T^a{}_b x^b - \frac{1}{r}\tilde{\mathcal{P}}_a T^a{}_b v^b \right) + \bar{\mathcal{C}}_a L^a{}_b v^b + \tilde{\mathcal{P}}_a L^a{}_b x^b \tag{6.36}$$

In particular with $T^a{}_b = M^a{}_b(x)$ and $L^a{}_b = 0$ (6.36) reduces exactly to (6.24) since $M^a{}_b(x)x^b = x^a$. Thus, we have showed that (6.24) is a coBRST charge. Notice that we equally well may choose ($T^a{}_b = \delta^a_b$, $L^a{}_b = 0$)

$${}^*Q = \lambda \left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r}\mathcal{P}_a (M^{-1})^a{}_b v^b \right) \tag{6.37}$$

In fact, the states (6.32) are invariant under (6.37) if $r = \pm 1/\alpha'$ and $r = \pm\beta'$ respectively, where α' and β' are given by (6.16) and (6.17) where α and β now are those in (6.33).

7 Summary and conclusions

In this paper we have considered gauge fixing and coBRST invariance of both abelian and nonabelian gauge theories. The gauge theories were given in standard BFV-form

and quantized on a state space V with a nondegenerate inner product $\langle u|v\rangle$. This inner product of V was required to be a linear form on a Hilbert space which means that V is a Krein space [18, 11]. This is a property which always allows us to define a coBRST charge. The metric operator η that relates V with a Hilbert space is expressed in terms of the indefinite oscillators in the theory and has the property $\eta^2 = \mathbf{1}$. The coBRST charge ${}^*Q$ is defined in terms of η and the nilpotent BRST charge Q by ${}^*Q \equiv \eta Q \eta$. The BRST singlets, $|s\rangle$, the states that represent the true physical degrees of freedom and which constitute a representation of the BRST cohomology ($|s\rangle \in \mathrm{Ker}Q/\mathrm{Im}Q$) are determined by the conditions

$$Q|s\rangle = {}^*Q|s\rangle = 0 \tag{7.1}$$

or equivalently

$$\triangle|s\rangle = 0, \quad \triangle \equiv [Q, {}^*Q]. \tag{7.2}$$

The questions we have tried to answer in this paper are the following ones: What is the general BFV-form of the coBRST charge ${}^*Q$ and what is the general form of the gauge fixing fermions ψ in the representations of BRST singlets found in [1], *i.e.* $|s\rangle = e^{[Q,\psi]}|\phi\rangle$ where $|\phi\rangle$ is a simple BRST invariant state? The answers to these two questions turned out to be interrelated since we have found that ψ may be chosen to be equal to a coBRST charge. Below we summarize our results and discuss their implications.

For the abelian models introduced in section 2 ($Q = \mathcal{C}^a p_a + \bar{\mathcal{P}}^a \pi_a$) we found in section 4 that the singlet states are determined by

$$\phi_a|s\rangle = \rho^a|s\rangle = k_a|s\rangle = \xi^a|s\rangle = 0 \tag{7.3}$$

or

$$\phi_a^\dagger|s\rangle = \rho^{a\dagger}|s\rangle = k_a^\dagger|s\rangle = \xi^{a\dagger}|s\rangle = 0 \tag{7.4}$$

where

$$\begin{aligned} \phi_a &\equiv \frac{1}{\sqrt{2}}(p_a + ir\pi_a), \quad \xi^a \equiv \frac{1}{\sqrt{2}}(ix^a - \frac{1}{r}v^a) \\ \rho^a &\equiv \frac{1}{\sqrt{2}}(\mathcal{C}^a + i\frac{1}{r}\bar{\mathcal{P}}^a), \quad k_a \equiv \frac{1}{\sqrt{2}}(\mathcal{P}_a + ir\bar{\mathcal{C}}_a) \end{aligned} \tag{7.5}$$

where in turn r is a real constant different from zero. Notice that the solutions of (7.3) and (7.4) constitute two different representations. Which one is realized depends on the choice of the original state space V. A given V will only allow for solutions of one of these conditions. One may notice that the two solutions correspond to solutions of (7.3) for opposite signs of r in (7.5). Solutions of (7.3) for different r's but with the same signs are unitarily equivalent. We have $|s\rangle_l' = U(\gamma)|s\rangle_l$ where γ is a real constant and where $U(\gamma) = U_1(\gamma)U_4(\gamma)$ or $U(\gamma) = U_2^\dagger(\gamma)U_3(\gamma)$ where in turn the unitary operators U_1, U_2, U_3, U_4 are defined in appendix B. $|s\rangle'$ satisfies then the same conditions as $|s\rangle$ with r replaced by re^γ.

In section 5 we determined the general form of the coBRST charge for the abelian models of section 2. The metric operator η was then expressed in terms of the indefinite oscillators in the theory which were identified by a diagonalization of the oscillators in

(7.5). We found then that ${}^*Q$ is not uniquely defined since η may be defined in several different ways even for one given r simply since the diagonalization of (7.5) is not unique. ${}^*Q$ for different signs of r's are related by ${}^*Q \rightarrow -{}^*Q$, and ${}^*Q$ for different r's but with the same signs are related by a unitary transformation of the form mentioned above. A simple form of ${}^*Q$ in terms of the original variables given in section 2 was found to be

$$ {}^*Q = \lambda\left(r\bar{\mathcal{C}}_a x^a - \frac{1}{r}\mathcal{P}_a v^a\right) \tag{7.6} $$

where λ is a real positive constant.

For the nonabelian models treated in section 6 we found essentially the same results. It is remarkable that although the oscillators in (7.5) then are nonlinear in the original variables the coBRST charge may still be of the same form as for abelian models. The general forms of coBRST found in section 4 suggests that the general BFV form of the coBRST charge is

$$ {}^*Q = \bar{\mathcal{C}}_a \chi^a - \mathcal{P}_a \Lambda^a \tag{7.7} $$

where χ^a and Λ^a are gauge fixing conditions to the gauge generators and the conjugate momenta to the Lagrange multipliers respectively. In fact, they must be related to the natural gauge fixing variables x^a and the Lagrange multipliers v^a by positive matrices. However, since the coBRST charge is nilpotent the form (7.7) requires the gauge conditions χ^a and Λ^a to be abelian. The most general nilpotent coBRST charge will allow for gauge conditions which are in involution. However, in this case there are nonlinear terms in the ghosts on the right-hand side of (7.7) (cf the construction of a nilpotent BRST charge [4, 5]).

We have investigated the properties of the representations

$$ |s\rangle_l = e^{[Q,\psi]}|\phi\rangle_l, \quad l = 1, 2 \tag{7.8} $$

for the singlet states found in [1] in the case when the gauge fixing fermion ψ has the form

$$ \psi = \alpha\mathcal{P}_a v^a + \beta\bar{\mathcal{C}}_a x^a \tag{7.9} $$

where α and β are real constants, and when $|\phi\rangle_l$ is chosen to satisfy the conditions in (2.6) or (6.18). We have then found that (see appendices A and C)

$$ |s\rangle_1 = e^{\alpha'[Q,\mathcal{P}_a v^a]}|\phi\rangle_1, \quad |s\rangle_2 = e^{\beta'[Q,\bar{\mathcal{C}}_a x^a]}|\phi\rangle_2 \tag{7.10} $$

where

$$ \alpha' = \alpha\,\frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}, \quad \beta' = \beta\,\frac{\tan\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}} \tag{7.11} $$

for $\alpha\beta > 0$ and

$$ \alpha' = \alpha\,\frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}}, \quad \beta' = \beta\,\frac{\tanh\sqrt{-\alpha\beta}}{\sqrt{-\alpha\beta}} \tag{7.12} $$

for $\alpha\beta < 0$. (The limit $\alpha\beta \rightarrow 0$ yields $\alpha' = \alpha$ and $\beta' = \beta$.) For abelian models it follows then from ref.[7] that $|s\rangle_1$ and $|s\rangle_2$ are inner product states if α' and β' are finite and non-zero

which in turn requires $\alpha \neq 0$ and $\beta \neq 0$ respectively together with $\sqrt{\alpha\beta} \neq n\pi/2$ for any positive integer n. Exactly under these conditions $|s\rangle_1$ and $|s\rangle_2$ are also singlet states. In fact, $|s\rangle_1$ and $|s\rangle_2$ satisfy the singlet conditions (7.3) if $r = 1/\alpha'$ and $r = -\beta'$ respectively. The results (7.10) shows that there are many formally different representations (7.8) which really are equal (*i.e.* many different α and β in (7.9) lead to the same α' and β' in (7.10)). Notice that both $|s\rangle_1$ and $|s\rangle_2$ in (7.10) cannot satisfy the singlet conditions (7.3) for a given r since this requires $r = 1/\alpha'$ and $r = -\beta'$ which implies $\alpha'\beta' = -1$ which has no solution. However, both $|s\rangle_1$ and $|s\rangle_2$ can be coBRST invariant under the same coBRST charge. Invariance under (7.6) for a given r requires $\alpha\beta > 0$ and $\tan\alpha\beta = 1$ *i.e.* $r = \pm\sqrt{\beta/\alpha}$. (These conditions follow from the fact that $r = \pm 1/\alpha'$ and $r = \pm\beta'$ allow for $\alpha'\beta' = 1$.) Essentially the same results were also found for the nonabelian models in section 6.

There are certainly still more involved forms for the gauge fixing fermions ψ than (7.9) which are allowed in (7.8). For the simple abelian theory we could *e.g.* consider $\psi = \mathcal{P}_a T^{ab} v^b + \bar{\mathcal{C}}_a S^{ab} x^b$ where T^{ab} and S^{ab} are real, invertible matrices. The analysis of this case is much more involved than the one of (7.9). One may notice that such a ψ is allowed for either $T^{ab} = 0$ or $S^{ab} = 0$. Furthermore, if T^{ab} and S^{ab} are symmetric and commuting one may prove that the representation (7.8) is a singlet state up similar conditions to the ones we have for (7.9) using exactly the same analysis we have used for (7.9). An example of such a gauge fixing is also considered for the nonabelian models in section 6. This suggests that even a gauge fixing fermion of the general BFV form (see *e.g.* [4]), *i.e.* $\psi = \bar{\mathcal{C}}_a \chi^a + \mathcal{P}_a \Lambda^a$ in representations like (7.8) do in fact yield singlet states.

One of the important results of our paper is that the coBRST charge is of the form of an allowed gauge fixing fermion. We may therefore replace ψ by a ${}^*Q$ in the representation (7.8) in which case we have

$$|s\rangle_l = e^{[Q,\, {}^*Q]}|\phi\rangle_l \equiv e^{\triangle}|\phi\rangle_l, \quad l = 1, 2 \tag{7.13}$$

Both for the abelian and nonabelian models our results show that the choice (7.6) of the coBRST charge always makes (7.13) a singlet state with a finite norm. However, one may notice that this singlet state is not coBRST invariant under the same ${}^*Q$ since $|\phi\rangle_l$ is never coBRST invariant by itself (${}^*Q$ commutes with $\triangle$). On the other hand, $|s\rangle_1$ ($|s\rangle_2$) in (7.13) is coBRST invariant under a different coBRST charge, ${}^*Q'$, obtained by the replacement $r \rightarrow \pm r/(\tanh\lambda)$ ($r \rightarrow \pm r \tanh\lambda$) in ${}^*Q$. Now ${}^*Q'$ and ${}^*Q$ are related by a unitary transformation involving a unitary operator of the last form in (B.8). This means that there are always unitary operators $U_{(l)}$ such that the singlet states

$$|s\rangle_l' \equiv U_{(l)} e^{[Q,\, {}^*Q]}|\phi\rangle_l, \quad l = 1, 2 \tag{7.14}$$

are invariant under ${}^*Q$. $U_{(1)}$ and $U_{(2)}$ may *e.g.* be chosen to be $U_2^\dagger(\gamma)U_3(\gamma)$ in appendix B with $\gamma = \ln(\tanh\gamma)$ and $\gamma = -\ln(\tanh\gamma)$ respectively.

A further intriguing feature of the representation (7.8) was discovered in sections 3 and 6. In the abelian case with (7.8) written as (see (3.11))

$$|s\rangle_l = e^{\alpha K_1 + \beta K_2}|\phi\rangle_l, \quad l = 1, 2 \tag{7.15}$$

where $K_1 \equiv [Q, \mathcal{P}_a v^a]$ and $K_2 \equiv [Q, \bar{\mathcal{C}}_a x^a]$, we found that K_1, K_2 and $K_3 = i[K_1, K_2]/2$ satisfy an $SL(2,R)$ algebra. This was also shown to be the case for the nonabelian models in section 6 for appropriate definitions of K_1 and K_2. Although this was not true for

the most natural generalization of K_1 and K_2 even these operators were shown to satisfy effectively an $SL(2,R)$ algebra, *i.e.* they satisfy an $SL(2,R)$ algebra on the states $|\phi\rangle_l$. Consequently the factor $e^{[Q,\psi]}$ may be viewed as a group transformation belonging to a one-dimensional subgroup of $SL(2,R)$. When α and β have the same signs it belongs to a compact subgroup while opposite signs of α and β makes it belong to a noncompact one. These two possibilities are quite different. In fact, there is a strong argument against the first possibility. One may notice that the connection between the representation (7.8) and the gauge fixing in the conventional BFV theory requires us in fact to identify our $[Q,\psi]$ with tH where H is a Hamiltonian operator given by $[Q,\psi']$ [9]. The proper identification of ψ and ψ' is therefore $\psi = t\psi'$. The replacement of α, β by $t\alpha$, $t\beta$ in (7.9) leads then to (7.10) with

$$\alpha' = \alpha\,\frac{\tan\left(|t|\sqrt{\alpha\beta}\right)}{\sqrt{\alpha\beta}}\,\mathrm{sign}\,t, \quad \beta' = \beta\,\frac{\tan\left(|t|\sqrt{\alpha\beta}\right)}{\sqrt{\alpha\beta}}\,\mathrm{sign}\,t \tag{7.16}$$

for $\alpha\beta > 0$ and

$$\alpha' = \alpha\,\frac{\tanh\left(|t|\sqrt{-\alpha\beta}\right)}{\sqrt{-\alpha\beta}}\,\mathrm{sign}\,t, \quad \beta' = \beta\,\frac{\tanh\left(|t|\sqrt{-\alpha\beta}\right)}{\sqrt{-\alpha\beta}}\,\mathrm{sign}\,t \tag{7.17}$$

for $\alpha\beta < 0$. Thus, if α and β have the same sign, $|s\rangle_1$ and $|s\rangle_2$ in (7.10) will be badly defined for infinitely many instants, $t = n\pi/(2\sqrt{\alpha\beta})$ where n is an integer, while opposite signs of α and β makes $t = 0$ the only badly defined instant. Remarkably enough the coBRST charge (7.6) belongs to the latter category and is therefore a good gauge fixing fermion even in this more restricted sense. In fact, our analysis indicates that any noncompact gauge choice ($\alpha\beta < 0$) may be represented by a coBRST charge.

From our results in sections 5 and 6 it is also possible to make use of a more general ${}^*Q$ than (7.6) in (7.13). In the general case the form (7.7) should be relevant as a gauge fixing fermion provided the gauge fixing variables χ^a and Λ^a are abelian. This form of ψ we have in *e.g.* QED and Yang-Mills. One may notice that essentially only abelian gauge fixing has been used in the literature so far. (This is *e.g.* required in the proof of gauge invariance given in [19].) It would certainly be interesting to understand what possible role the nonlinear terms in the coBRST charge ${}^*Q$ for nonabelian gauge fixing can possibly play when ${}^*Q$ is viewed as a gauge fixing fermion. Anyway, apart from this question mark, our results suggest that the coBRST charge ${}^*Q$ is always a good gauge fixing fermion and a candidate for a natural choice of ψ.

We end with a comment on the difference between coBRST and antiBRST. These two concepts are often confused in the literature. Like the coBRST charge (${}^*Q$) also the antiBRST charge ($\bar{Q}$) has ghost number minus one and is nilpotent. However, contrary to the coBRST charge the antiBRST charge anticommutes with the BRST charge and is a symmetry of the model. For the simple abelian model in section 2 the antiBRST charge is given by [20, 21]

$$\bar{Q} = p_a\bar{\mathcal{C}}^a - \mathcal{P}_a\pi^a = [Q, \mathcal{P}_a\bar{\mathcal{C}}^a] \tag{7.18}$$

In this case the coBRST charge (7.6) may be expressed in terms of $\bar{Q}$ as follows

$${}^*Q = i[\bar{Q}, S] \tag{7.19}$$

where

$$S \equiv \frac{\lambda}{2}\left(r x_a x^a + \frac{1}{r} v_a v^a\right) \tag{7.20}$$

The form (7.19) of $^*Q$ is also the form of a gauge fixing fermion in antiBRST invariant theories (cf. [21]).

There are also other charges in the literature which have ghost number minus one and which are nilpotent. The proposal of coBRST in [1] for the abelian model was $Q' = \bar{\mathcal{C}}_a v^a + \mathcal{P}_a x^a$ which differs from (7.8) (it yields zero on $|\phi\rangle_l$). In [22, 23, 24] a Q' is defined by exchanging all ghosts with their conjugate momenta which implies $Q' = p_a \mathcal{P}^a + \bar{\mathcal{C}}_a \pi^a$ for the abelian model. (This Q' was called antiBRST in [23] and coBRST in [24].)

Appendix A

Proof of eq.(3.16)

Making use of formulas (B.20)-(B.30) in appendix B of ref.[16] we find the following equalities

$$e^{\alpha K_1+\beta K_2}=e^{\gamma K_2}e^{\delta K_1}e^{\gamma K_2}=e^{\gamma' K_1}e^{\delta' K_2}e^{\gamma' K_1} \tag{A.1}$$

where the parameters γ,δ,γ' and δ' are given by

$$\begin{aligned}
&\gamma=\frac{\sqrt{\alpha\beta}}{\alpha}\tan\frac{1}{2}\sqrt{\alpha\beta},\quad \delta=\frac{\alpha}{\sqrt{\alpha\beta}}\sin\sqrt{\alpha\beta}\\
&\gamma'=\frac{\sqrt{\alpha\beta}}{\beta}\tan\frac{1}{2}\sqrt{\alpha\beta},\quad \delta'=\frac{\beta}{\sqrt{\alpha\beta}}\sin\sqrt{\alpha\beta}\\
&\cos\sqrt{\alpha\beta}=1-\delta\gamma=1-\delta'\gamma'
\end{aligned} \tag{A.2}$$

for $\alpha\beta>0$. For $\alpha\beta<0$ we have the same relations with the replacement (3.7).

By means of (A.1) and

$$K_2|\phi\rangle_1=K_1|\phi\rangle_2=K_3|\phi\rangle_1=K_3|\phi\rangle_2=0 \tag{A.3}$$

we get therefore

$$\begin{aligned}
|s\rangle_1&=e^{\alpha K_1+\beta K_2}|\phi\rangle_1=e^{\gamma K_2}e^{\delta K_1}|\phi\rangle_1=\sum_{n,m=0}^{\infty}\frac{\gamma^n\delta^m}{n!m!}K_2^nK_1^m|\phi\rangle_1,\\
|s\rangle_2&=e^{\alpha K_1+\beta K_2}|\phi\rangle_2=e^{\gamma' K_1}e^{\delta' K_2}|\phi\rangle_2=\sum_{n,m=0}^{\infty}\frac{\gamma'^n\delta'^m}{n!m!}K_1^nK_2^m|\phi\rangle_2
\end{aligned} \tag{A.4}$$

It is easily seen from the algebra (3.14) that

$$\begin{aligned}
&[K_1,K_2^n]=-2inK_2^{n-1}K_3+n(n-1)K_2^{n-1},\\
&[K_2,K_1^n]=2inK_1^{n-1}K_3+n(n-1)K_1^{n-1}
\end{aligned} \tag{A.5}$$

Hence, due to (A.3) we have

$$K_2K_1^n|\phi\rangle_1=n(n-1)K_1^{n-1}|\phi\rangle_1,\quad K_1K_2^n|\phi\rangle_2=n(n-1)K_2^{n-1}|\phi\rangle_2 \tag{A.6}$$

which implies

$$\begin{aligned}
K_2^mK_1^n|\phi\rangle_1&=n(n-1)^2(n-2)^2\cdots(n-m+1)^2(n-m)K_1^{n-m}|\phi\rangle_1,\\
K_1^mK_2^n|\phi\rangle_2&=n(n-1)^2(n-2)^2\cdots(n-m+1)^2(n-m)K_2^{n-m}|\phi\rangle_2
\end{aligned} \tag{A.7}$$

When this is inserted into (A.4) we find therefore

$$|s\rangle_1=\exp\left(\frac{\delta}{1-\delta\gamma}K_1\right)|\phi\rangle_1,\quad |s\rangle_2=\exp\left(\frac{\delta'}{1-\delta'\gamma'}K_2\right)|\phi\rangle_2 \tag{A.8}$$

where

$$\frac{\delta}{1-\delta\gamma}=\frac{\alpha}{\sqrt{\alpha\beta}}\tan\sqrt{\alpha\beta},\quad \frac{\delta'}{1-\delta'\gamma'}=\frac{\beta}{\sqrt{\alpha\beta}}\tan\sqrt{\alpha\beta} \tag{A.9}$$

Appendix B

Some unitary symmetries.

Let us introduce the following unitary operators for the abelian model introduced in section 2

$$
\begin{aligned}
&U_1(\gamma)\equiv\exp\{i\gamma\frac{1}{2}(p_ax^a+x^ap_a)\},\quad U_2(\gamma)\equiv\exp\{i\gamma\frac{1}{2}(\pi_av^a+v^a\pi_a)\}\\
&U_3(\gamma)\equiv\exp\{\gamma\frac{1}{2}(\mathcal{C}^a\mathcal{P}_a-\mathcal{P}_a\mathcal{C}^a)\},\quad U_4(\gamma)\equiv\exp\{\gamma\frac{1}{2}(\bar{\mathcal{C}}_a\bar{\mathcal{P}}^a-\bar{\mathcal{P}}^a\bar{\mathcal{C}}_a)\}
\end{aligned}
\tag{B.1}
$$

where γ is a real constant. These operators act as scaling operators on the original variables. The nontrivial transformations are

$$
\begin{aligned}
&U_1(\gamma)x^aU_1^\dagger(\gamma)=e^{\gamma}x^a,\quad U_1(\gamma)p_aU_1^\dagger(\gamma)=e^{-\gamma}p_a\\
&U_2(\gamma)v^aU_2^\dagger(\gamma)=e^{\gamma}v^a,\quad U_2(\gamma)\pi_aU_2^\dagger(\gamma)=e^{-\gamma}\pi_a\\
&U_3(\gamma)\mathcal{C}^aU_3^\dagger(\gamma)=e^{\gamma}\mathcal{C}^a,\quad U_3(\gamma)\mathcal{P}_aU_3^\dagger(\gamma)=e^{-\gamma}\mathcal{P}_a\\
&U_4(\gamma)\bar{\mathcal{C}}_aU_4^\dagger(\gamma)=e^{\gamma}\bar{\mathcal{C}}_a,\quad U_4(\gamma)\bar{\mathcal{P}}^aU_4^\dagger(\gamma)=e^{-\gamma}\bar{\mathcal{P}}^a
\end{aligned}
\tag{B.2}
$$

Consider then the representation

$$
|s\rangle=e^{[Q,\psi]}|\phi\rangle \tag{B.3}
$$

where $|\phi\rangle$ satisfies the properties of $|\phi\rangle_1$ or $|\phi\rangle_2$ in (2.6). Under the unitary transformations (B.1) they satisfy ($a=1,2,\ldots,n$)

$$
\begin{aligned}
&U_1(\gamma)|\phi\rangle_1=e^{-\frac{1}{2}n\gamma}|\phi\rangle_1,\quad U_2(\gamma)|\phi\rangle_1=e^{\frac{1}{2}n\gamma}|\phi\rangle_1\\
&U_3(\gamma)|\phi\rangle_1=e^{\frac{1}{2}n\gamma}|\phi\rangle_1,\quad U_4(\gamma)|\phi\rangle_1=e^{\frac{1}{2}n\gamma}|\phi\rangle_1\\
&U_1(\gamma)|\phi\rangle_2=e^{\frac{1}{2}n\gamma}|\phi\rangle_2,\quad U_2(\gamma)|\phi\rangle_2=e^{-\frac{1}{2}n\gamma}|\phi\rangle_2\\
&U_3(\gamma)|\phi\rangle_2=e^{-\frac{1}{2}n\gamma}|\phi\rangle_2,\quad U_4(\gamma)|\phi\rangle_2=e^{-\frac{1}{2}n\gamma}|\phi\rangle_2
\end{aligned}
\tag{B.4}
$$

which is another sign of the fact that $|\phi\rangle$ is *not* an inner product state. We notice now that the following combinations of the unitary operators (B.1) leave the BRST charge (2.1) invariant:

$$
\begin{aligned}
&U_1(\gamma)U_3(\gamma),\quad U_2(\gamma)U_4^\dagger(\gamma),\quad U_1(\gamma)U_2(\gamma)U_3(\gamma)U_4^\dagger(\gamma)\\
&U_1(\gamma)U_2^\dagger(\gamma)U_3(\gamma)U_4(\gamma)
\end{aligned}
\tag{B.5}
$$

and the following combinations scale Q:

$$
\begin{aligned}
&U_1(\gamma)U_2(\gamma),\quad U_3^\dagger(\gamma)U_4(\gamma),\quad U_1(\gamma)U_4(\gamma),\quad U_2(\gamma)U_3^\dagger(\gamma)\\
&U_1(\gamma)U_2(\gamma)U_3^\dagger(\gamma)U_4(\gamma)
\end{aligned}
\tag{B.6}
$$

(The first four combinations yield $Q\rightarrow e^{-\gamma}Q$ and the last $Q\rightarrow e^{-2\gamma}Q$.) All combinations in (B.5) and (B.6) yield unity on $|\phi\rangle$ for any of the two sets of conditions in (2.6).

Let now U be any of the combinations in (B.5) and (B.6). We find then for the representation (B.3) with $\psi=\alpha\mathcal{P}_av^a+\beta\bar{\mathcal{C}}_ax^a$:

$$
U|s\rangle=e^{[Q,\psi']}|\phi\rangle \tag{B.7}
$$

where

$$\psi' = \psi \quad \text{for} \quad U = U_1(\gamma)U_2(\gamma),\ U_3(\gamma)U_4^\dagger(\gamma)$$
$$\psi' = \alpha e^{-\gamma}\mathcal{P}_a v^a + \beta e^{\gamma}\bar{\mathcal{C}}_a x^a \quad \text{for} \quad U = U_1(\gamma)U_4(\gamma),\ U_2^\dagger(\gamma)U_3(\gamma) \tag{B.8}$$

Notice that the generator of $U_3(\gamma)U_4^\dagger(\gamma)$ $(U_1(\gamma)U_2(\gamma))$ is the ghost number operator when we have fermionic (bosonic) ghosts. $|s\rangle$ is invariant under these combinations since $|s\rangle$ has ghost number zero.

For the nonabelian models of section 6 we still have that $U_2(\gamma)U_4^\dagger(\gamma)$ leaves the BRST charge (6.1) invariant, and that $U_3^\dagger(\gamma)U_4(\gamma)$ and $U_2(\gamma)U_3^\dagger(\gamma)$ scale it $(Q \rightarrow e^{-\gamma}Q)$. Thus, (B.7) is still valid with $\psi' = \psi$ for $U_3(\gamma)U_4^\dagger(\gamma)$ and $\psi' = \alpha e^{-\gamma}\mathcal{P}_a v^a + \beta e^{\gamma}\bar{\mathcal{C}}_a x^a$ for $U_2^\dagger(\gamma)U_3(\gamma)$. Notice that the ghost number operator is also here the generator of $U_3(\gamma)U_4^\dagger(\gamma)$.

Appendix C

Some properties used in section 6.

In section 6 the gauge fixing variables x^a satisfy

$$[x^a, x^b] = 0, \quad [x^a, \theta_b] = iM^a{}_b \tag{C.1}$$

where $M^a{}_b$ is an invertible matrix operator. If x^a are chosen to be canonical group coordinates then $M^a{}_b$ depends only on x^a and satisfies the equations

$$(\partial_d M^c{}_a)M^d{}_b - (\partial_d M^c{}_b)M^d{}_a = U_{ab}{}^d M^c{}_d \tag{C.2}$$

where the derivatives are with respect to x^a. These are the equations for the vielbeins of the group. The solution may be obtained as a power series in x^a. To the first orders we have

$$\begin{aligned} M^a_b = \quad & \delta^a_b + \frac{1}{2}U_{bc}{}^a x^c + \frac{1}{12}U_{be_1}{}^d U_{de_2}{}^a x^{e_1}x^{e_2} - \\ & -\frac{1}{720}U_{be_1}{}^{d_1}U_{d_1e_2}{}^{d_2}U_{d_2e_3}{}^{d_3}U_{d_3e_4}{}^a x^{e_1}x^{e_2}x^{e_3}x^{e_4} + O(x^6) \end{aligned} \tag{C.3}$$

The inverse of M is of the particularly simple form:

$$\begin{aligned} (M^{-1})^a{}_b = \quad & \delta^a_b - \frac{1}{2!}U_{bc}{}^a x^c + \frac{1}{3!}U_{be_1}{}^d U_{de_2}{}^a x^{e_1}x^{e_2} - \\ & -\frac{1}{4!}U_{be_1}{}^{d_1}U_{d_1e_2}{}^{d_2}U_{d_2e_3}{}^a x^{e_1}x^{e_2}x^{e_3} + \\ & +\frac{1}{5!}U_{be_1}{}^{d_1}U_{d_1e_2}{}^{d_2}U_{d_2e_3}{}^{d_3}U_{d_3e_4}{}^a x^{e_1}x^{e_2}x^{e_3}x^{e_4} + O(x^5) \end{aligned} \tag{C.4}$$

In terms of this $M^a{}_b$ the hermitian gauge generators θ_a may be represented as

$$\theta_a = \frac{1}{2}\left(p_b M^b{}_a + M^b{}_a p_b\right) \tag{C.5}$$

where p_a are hermitian conjugate momenta to x^a. From (C.1) and (C.5) we have

$$p_a = (M^{-1})^b{}_a \theta_b + i\frac{1}{2}(M^{-1})^b{}_a \partial_c M^c{}_b, \quad [x^a, p_b] = i\delta^a_b \tag{C.6}$$

Notice that (C.2) implies $[\theta_a, \theta_b] = iU_{ab}{}^c\theta_c$. Another property of $M^a{}_b$ is

$$M^a{}_b x^b = (M^{-1})^a{}_b x^b = x^a \tag{C.7}$$

Proof of (6.14):
Using the form (6.10) of K_1 and K_2 as well as the definition (6.11) of K_3 we have

$$[K_2, K_3] = -i\pi_a M^a{}_b x^b + \bar{\mathcal{C}}_a M^a{}_b M^b{}_c \mathcal{C}^c + i\bar{\mathcal{C}}_a \mathcal{C}^b x^c [\theta_b, M^a{}_c] \tag{C.8}$$

where we have made use of the general Jacobi identities (6.5). Now from (C.7) and

$$M^a{}_b = i[\theta_b, M^a{}_c]x^c + M^a{}_c M^c{}_b \tag{C.9}$$

which follows from (C.7) (use $[\theta_b, M^a{}_c x^c - x^a] = 0$) we find

$$[K_2, K_3] = iK_2 \tag{C.10}$$

Similarly we find straight-forwardly

$$\begin{aligned}
[K_1, K_3] &= i(\theta_a + \theta^{gh}_a)\left(M^a{}_b + \frac{1}{2}U_{cb}{}^a x^c\right)v^b - \mathcal{P}_a\left(M^a{}_b + \frac{1}{2}U_{cb}{}^a x^c\right)v^b + \\
&+\frac{1}{2}\pi_a[\theta_d, M^a{}_b]v^b v^d + \mathcal{P}_a\mathcal{C}^b v^c\left(\frac{1}{2}U_{cd}{}^a M^d{}_b - i[\theta_b, M^a{}_c]\right) + \\
&+i\frac{1}{2}\bar{\mathcal{C}}_a\bar{\mathcal{P}}^b v^c\left([\theta_b, M^a{}_c] + [\theta_c, M^a{}_b]\right) - \frac{1}{2}\bar{\mathcal{C}}_a\mathcal{C}^b v^c v^d[\theta_b, [\theta_d, M^a{}_c]]
\end{aligned} \tag{C.11}$$

where we have made use of the general Jacobi identities (6.12) and

$$U_{db}{}^e[\theta_e, M^a{}_c] = i[\theta_b, [\theta_d, M^a{}_c]] - i[\theta_d, [\theta_b, M^a{}_c]] \tag{C.12}$$

We notice now that if x^a are chosen as canonical coordinates on the group manifold then we have

$$\left(M^a{}_b + \frac{1}{2}U_{cb}{}^a x^c\right)|\phi\rangle_1 = \delta^a_b|\phi\rangle_1, \quad \left([\theta_b, M^a{}_b] + [\theta_c, M^a{}_b]\right)|\phi\rangle_1 = 0 \tag{C.13}$$

and

$$[K_1, K_3]|\phi\rangle_1 = -iK_1|\phi\rangle_1 \tag{C.14}$$

This proves eq.(6.14)•

Eq.(C.10) is easily seen to imply

$$[K_1, K_2^n] = -2inK_2^{n-1}K_3 + n(n-1)K_2^{n-1} \tag{C.15}$$

Hence, we have

$$K_1 K_2^n|\phi\rangle_2 = n(n-1)K_2^{n-1}|\phi\rangle_2 \tag{C.16}$$

We also expect

$$K_2 K_1^n|\phi\rangle_1 = n(n-1)K_1^{n-1}|\phi\rangle_1 \tag{C.17}$$

This requires apart from (C.14) also

$$[K_1, [K_1, K_3]]|\phi\rangle_1 = [K_1, [K_1, [K_1, K_3]]]|\phi\rangle_1 = \ldots = 0 \tag{C.18}$$

This we have checked to lowest order and the structure of (C.11) seems to make it true for any order. However, we have no rigorous proof. Anyway we feel rather confident that (C.17) is valid for any n. Eqs. (C.16) and (C.17) imply (A.7) and the formulas (A.8) in appendix A. (Notice that the formulas (A.1) were not necessary for the derivation of (A.8). They only provided for a convenient way to obtain the nice expressions of the coefficients in the expansions (A.4).)

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